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The natural affinors on generalized higher order tangent bundles

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Riassunto: *Per interi $r \geq 1$ e $n \geq 2$ e un numero reale $a < 0$, si classificano tutti gli "affinors" naturali su $T^{(r),a}$ fascio generalizzato tangente di ordine più alto e su $E^{(r),a}$ fascio generalizzato tangente di ordine più alto esteso a varietà n-dimensionali.*

Abstract: *For integers $r \geq 1$ and $n \geq 2$ and a real number $a < 0$ the natural affinors on the generalized higher order tangent bundle $T^{(r),a}$ and on the extended generalized higher order tangent bundle $E^{(r),a}$ over n-manifolds are classified.*

0 – Let us recall the following definitions (see for ex. [2]).

Let $F : \mathcal{M}f_n \to \mathcal{FM}$ be a functor from the category $\mathcal{M}f_n$ of all n-dimensional manifolds and their local diffeomorphisms into the category $\mathcal{FM}$ of fibered manifolds. Let B be the base functor from the category of fibered manifolds to the category of manifolds.

A *natural bundle* over n-manifolds is a functor F satisfying $B \circ F = \mathrm{id}$ and the localization condition: for every inclusion of an open subset $i_U : U \to M$, FU is the restriction $p_M^{-1}(U)$ of $p_M : FM \to M$ over U and Fi_U is the inclusion $p_M^{-1}(U) \to FM$.

An *affinor* Q on a manifold M is a tensor type $(1, 1)$, i.e. a linear

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morphism $Q : TM \to TM$ over id_M.

A *natural affinor* on a natural bundle F is a system of affinors $Q_M : TFM \to TFM$ on FM for every n-manifold M satisfying $TFf \circ Q_M = Q_N \circ TFf$ for every local diffeomorphism $f : M \to N$.

A *connection* on a fibre bundle Y is an affinor $\Gamma : TY \to TY$ on Y such that $\Gamma \circ \Gamma = \Gamma$ and $\mathrm{im}(\Gamma) = VY$, the vertical bundle of Y.

A *natural connection* on a natural bundle F is a system of connections $\Gamma_M : TFM \to TFM$ on FM for every n-manifold M which is (additionally) a natural affinor on F.

In [3] it was shown how natural affinors Q on some natural bundles FM can be used to study the torsion $\tau = [\Gamma, Q]$ of connections Γ on the same bundles FM. That is why, natural affinors have been classified in many papers. For example, in [1] natural affinors on the r-th order vector tangent bundle $T^{(r)}M = (J^r(M, \mathbb{R})_0)^*$ and on the extended r-th order vector tangent bundle $E^{(r)}M = (J^r(M, \mathbb{R}))^*$ were classified.

In this paper one considers the functor $F = T^{(r),a}$ ($a \in \mathbb{R}$) which associates to any n-manifold M its *generalized higher order tangent bundle* $T^{(r),a}M$. (This notion was introduced in [4]. This notion will be cited in Sect. 1.)

For integers $r \geq 1$ and $n \geq 2$ and a negative real number $a < 0$ we classify the natural affinors on $T^{(r),a}M$ and on the extended generalized higher order tangent bundle $E^{(r),a}M$. We prove that in both cases every natural affinor A is proportional to the identity affinor. (The similar result for $T^{(r),0} = T^{(r)}$ and $E^{(r),0} = E^{(r)}$ do not hold, [1].)

The above result shows that "torsion" of a connection Γ on $T^{(r),a}M$ or on $E^{(r),a}M$ makes no sense because of $[\Gamma, \mathrm{id}] = 0$. (In the case $a = 0$ the situation is different because there is a natural affinor which is not proportional to the identity affinor.)

The above result also shows that there are no natural connections on $T^{(r),a}$ and $E^{(r),a}$. (If $a = 0$, the same result hold because of [1].)

The usual coordinates on $\mathbb{R}^n$ are denoted by x^i and $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, \ldots, n$.

All manifolds and maps are assumed to be of class C^∞.

1 – Let us cite the notion of $T^{(r),a}M$, [4].

The linear action $\alpha^{(a)} : GL(n, \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$, $\alpha^{(a)}(B, x) = |\det(B)|^a x$

defines the natural vector bundle $T^{(0,0),a}M = LM \times_{\alpha(a)} \mathbb{R}$ (associated to the principal bundle LM of linear frames). Every embedding $\varphi : M \to N$ of n-manifolds induces a vector bundle mapping $T^{(0,0),a}\varphi = L\varphi \times_{\alpha(a)} \mathrm{id}_{\mathbb{R}} : T^{(0,0),a}M \to T^{(0,0),a}N$. Let $T^{r*,a}M = \{j^r_x\sigma \mid \sigma$ is a local section of $T^{(0,0),a}M$, $\sigma(x) = 0, x \in M\}$ be the vector bundle over M of all r-jets of local sections of $T^{(0,0),a}M$ with target 0 with respect to the source projection. We set $T^{(r),a}M = (T^{r*,a}M)^*$, the dual vector bundle. Every embedding $\varphi : M \to N$ of n-manifolds induces a vector bundle mapping $T^{r*,a}\varphi : T^{r*,a}M \to T^{r*,a}N$, $j^r_x\sigma \to j^r_{\varphi(x)}(T^{(0,0),a}\varphi \circ \sigma \circ \varphi^{-1})$, and (next) it induces a vector bundle mapping $T^{(r),a}\varphi = ((T^{r*,a}\varphi)^*)^{-1} : T^{(r),a}M \to T^{(r),a}N$ over φ, and we obtain a natural vector bundle $T^{(r),a}$ over n-manifolds. (For $a = 0$ we get the r-th order vector tangent bundle $T^{(r)}$. That is why $T^{(r),a}M$ is called the generalized higher order tangent bundle.)

Similarly, considering $J^rT^{(0,0),a}M = \{j^r_x\sigma | \sigma \text{ is a loc. sect. of } T^{(0,0),a}M\}$ instead of $T^{r*,a}M$ we get the extended generalized higher order tangent bundle $E^{(r),a}M = (J^rT^{(0,0),a}M)^*$, and we obtain a natural vector bundle $E^{(r),a}$ over n-manifolds. (For $a = 0$ we get the extended r-th order vector tangent bundle $E^{(r)} = T^{(r)} \times \mathbb{R}$.)

$T^{(0,0),a}M$ is the well-known bundle of densities with weight a. $E^{(r),a}M$ appears if we consider linear differential operators $D \in \mathrm{Dif}f^r(C^\infty_x(T^{(0,0),a}M), \mathbb{R})$ of order $\le r$ on the $C^\infty_x(M)$-module $C^\infty_x(T^{(0,0),a}M)$ of germs at $x \in M$ of fields of densities on M with weight a. These operators are in bijection with elements $I(D) \in E^{(r),a}_x M$. This bijection is given by $I(D)(j^r_x\sigma) = D(\mathrm{germ}_x(\sigma))$, σ is a field of densities of weight a on M. Thus $E^{(r),a}M$ is the vector bundle of such operators. $T^{(r),a}M \subset E^{(r),a}M$ is a vector subbundle. (The inclusion is the vector bundle map dual to the obvious jet projection.)

2 – In this section we study linear natural transformations $C : TT^{(r),a} \to T^{(r),a}$ over n-manifolds.

The linearity of C means that for any n-manifold M transformation C determines a linear map $T_y(T^{(r),a}M) \to T^{(r),a}_x M$ for $y \in T^{(r),a}_x M$, $x \in M$.

PROPOSITION 1. *For natural numbers r and $n \ge 2$ and a real number $a < 0$ every linear natural transformation $C : TT^{(r),a} \to T^{(r),a}$ over n-manifolds is* 0.

PROOF. From now on the set of all $\alpha \in (\mathbb{N} \cup \{0\})^n$ with $1 \le |\alpha| \le r$ will be denoted by $P(r,n)$.

Clearly, every section of $T^{(0,0),a}\mathbb{R}^n = L\mathbb{R}^n \times_{\alpha(a)} \mathbb{R}$ can be considered as a real valued function f on $\mathbb{R}^n$ satisfying the transformation rule $\varphi_* f(x) = |\det(d_0(\tau_{-x} \circ \varphi \circ \tau_{\varphi^{-1}(x)}))|^a \cdot f \circ \varphi^{-1}(x)$ for every local diffeomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$, where $\tau_y : \mathbb{R}^n \to \mathbb{R}^n$ is the translation by $y \in \mathbb{R}^n$. Then any element v from the fibre $T_0^{(r),a}\mathbb{R}^n$ of $T^{(r),a}\mathbb{R}^n$ over 0 is a linear combination of the $(j_0^r x^\alpha)^*$ for all $\alpha \in P(r,n)$, where the $(j_0^r x^\alpha)^*$ form the basis dual to the basis $j_0^r x^\alpha \in T_0^{r*,a}\mathbb{R}^n$. From now on we denote the coefficient of v corresponding to $(j_0^r x^\alpha)^*$ by $[v]_\alpha$.

Of course, any natural transformation C as above is (fully) determined by the contractions $\langle C(u), j_0^r x^\alpha\rangle \in \mathbb{R}$ for $u \in (TT^{(r),a})_0\mathbb{R}^n = \mathbb{R}^n \times (VT^{(r),a})_0\mathbb{R}^n = \mathbb{R}^n \times T_0^{(r),a}\mathbb{R}^n \times T_0^{(r),a}\mathbb{R}^n$ and $\alpha \in P(r,n)$, $j_0^r x^\alpha \in T_0^{r*,a}\mathbb{R}^n$.

Using similar method as in [4], we prove that C is fully determined by the values $\langle C(u), j_0^r(x^1)\rangle \in \mathbb{R}$ for $u \in (TT^{(r),a})_0\mathbb{R}^n$, where $j_0^r(x^1) \in T_0^{r*,a}\mathbb{R}^n$.

If $\alpha = (\alpha_1, \dots, \alpha_n) \in P(r,n)$ with $\alpha_1 + \dots + \alpha_{n-1} \ge 1$ and $\tau \in \mathbb{R}$, then the diffeomorphism $\varphi_{\alpha,\tau} = (x^1, \dots, x^{n-1}, x^n - \tau(x^1)^{\alpha_1} \cdot \ldots \cdot (x^{n-1})^{\alpha_{n-1}})$ sends $j_0^r((x^n)^{\alpha_n+1}) \in T_0^{r*,a}\mathbb{R}^n$ into $j_0^r((x^n + \tau(x^1)^{\alpha_1} \cdot \ldots \cdot (x^{n-1})^{\alpha_{n-1}})^{\alpha_n+1})$ (as $\varphi_{\alpha,\tau}^{-1} = (x^1, \dots, x^{n-1}, x^n + \tau(x^1)^{\alpha_1} \cdot \ldots \cdot (x^{n-1})^{\alpha_{n-1}})$ and $\det(d_0(\tau_{-\varphi_{\alpha,\tau}(y)} \circ \varphi_{\alpha,\tau} \circ \tau_y)) = 1$ for any $y \in \mathbb{R}^n$). Then by the naturality of C with respect to the diffeomorphisms $\varphi_{\alpha,\tau}$, the values $\langle C(u), j_0^r((x^n + \tau(x^1)^{\alpha_1} \cdot \ldots \cdot (x^{n-1})^{\alpha_{n-1}})^{\alpha_n+1})\rangle$ for $u \in (TT^{(r),a})_0\mathbb{R}^n$ and $\tau \in \mathbb{R}$ are determined by the values $\langle C(u), j_0^r((x^n)^{\alpha_n+1})\rangle$ for $u \in (TT^{(r),a})_0\mathbb{R}^n$. On the other hand, given $u \in (TT^{(r),a})_0\mathbb{R}^n$ the value $\frac{1}{\alpha_n+1}\langle C(u), j_0^r x^\alpha\rangle$ is the coefficient on τ of the polynomial $\langle C(u), j_0^r((x^n + \tau(x^1)^{\alpha_1} \cdot \ldots \cdot (x^{n-1})^{\alpha_{n-1}})^{\alpha_n+1})\rangle$ with respect to τ. Therefore the values $\langle C(u), j_0^r x^\alpha\rangle$ for $u \in (TT^{(r),a})_0\mathbb{R}^n$ are determined by the values $\langle C(u), j_0^r((x^n)^{\alpha_n+1})\rangle$ for $u \in (TT^{(r),a})_0\mathbb{R}^n$. Then C is fully determined by the values $\langle C(u), j_0^r((x^n)^i)\rangle$ for $u \in (TT^{(r),a})_0\mathbb{R}^n$ and $i = 1, \dots, r$. For $i \in \{1, \dots, r\}$ the diffeomorphism $\varphi_i = (x^1 - (x^n)^i, x^2, \dots, x^n)$ sends $j_0^r(x^1)$ into $j_0^r(x^1 + (x^n)^i)$ (as $\varphi_i^{-1} = (x^1 + (x^n)^i, x^2, \dots, x^n)$ and $\det(d_0(\tau_{-\varphi_i(y)} \circ \varphi_i \circ \tau_y)) = 1$ for any $y \in \mathbb{R}^n$). Then by the naturality of C with respect to φ_i, the values $\langle C(u), j_0^r((x^n)^i)\rangle$ for $u \in (TT^{(r),a})_0\mathbb{R}^n$ are fully determined by the values $\langle C(u), j_0^r(x^1)\rangle$ for $u \in (TT^{(r),a})_0\mathbb{R}^n$. That is why C is fully determined by the values $\langle C(u), j_0^r(x^1)\rangle \in \mathbb{R}$ for $u \in (TT^{(r),a})_0\mathbb{R}^n = \mathbb{R}^n \times T_0^{(r),a}\mathbb{R}^n \times T_0^{(r),a}\mathbb{R}^n$.

We continue the proof of the proposition. For any $t \in \mathbb{R}_+$ and any $\alpha \in P(r,n)$ the homothety $a_t = (tx^1, \dots, tx^n)$ sends $j^r_0 x^\alpha \in T^{r*,a}_0\mathbb{R}^n$ into $t^{na-|\alpha|} j^r_0 x^\alpha$, i.e. $(j^r_0 x^\alpha)^*$ into $t^{|\alpha|-na} \cdot (j^r_0 x^\alpha)^*$. Then (since $a < 0$) by the naturality of C with respect to a_t and the homogeneous function theorem [2] we deduce that given $u = (u_1, u_2, u_3) \in (TT^{(r),a})_0\mathbb{R}^n = \mathbb{R}^n \times T^{(r),a}_0\mathbb{R}^n \times T^{(r),a}_0\mathbb{R}^n$, $u_1 = (u^1_1, \dots, u^n_1) \in \mathbb{R}^n$, $u_2, u_3 \in T^{(r),a}_0\mathbb{R}^n$ we have $\langle C(u), j^r_0(x^1)\rangle = \sum_{i=1}^n \lambda_i [u_2]_{e_i} + \sum_{i=1}^n \mu_i [u_3]_{e_i} + \dots$, where λ_i, μ_i are the reals, the dots denote the linear combination of monomials in $u^1_1, \dots, u^n_1$ of degree ≥ 2 and $e_i = (0, \dots, 1, \dots, 0) \in P(r,n)$, 1 in i-th position.

Since C is linear, $\langle C(u), j^r_0(x^1)\rangle$ depends linearly on (u_1, u_3) for any u_2. Then $\langle C(u), j^r_0(x^1)\rangle = \sum_{i=1}^n \mu_i [u_3]_{e_i}$ for the reals μ_i.

For any $t \in \mathbb{R}_+$ the homothety $b_t = (x^1, tx^2, \dots, tx^n)$ sends $(j^r_0(x^i))^* \in T^{r*,a}_0\mathbb{R}^n$ into $t^{1-(n-1)a}(j^r_0(x^i))^*$ for $i = 2, \dots, n$, and it sends $(j^r_0(x^1))^*$ into $t^{-(n-1)a}(j^r_0(x^1))^*$. Then, by the naturality of C with respect to b_t and $a < 0$,

$$(*) \qquad \langle C(u), j^r_0(x^1)\rangle = \mu [u_3]_{e_1}$$

for the real number $\mu = \mu_1$. In particular, if $n \geq 2$

$$(**) \qquad \langle C(\partial^C_{2\,|\omega}), j^r_0(x^1)\rangle = \langle C(e_2, \omega, 0), j^r_0(x^1)\rangle = 0$$

for any $\omega \in T^{(r),a}_0\mathbb{R}^n$, where $(\ )^C$ is the complete lift of vector fields to $T^{(r),a}$.

Clearly, the proof of the proposition will be complete after proving that $\mu = 0$, i.e. $\langle C(0, 0, (j^r_0(x^1))^*), j^r_0(x^1)\rangle = 0$. But (if $n \geq 2$) we have

$$(***) \qquad \begin{aligned} 0 &= \langle C(((x^2)^r \partial_1)^C{}_{|\omega}), j^r_0(x^1)\rangle = \\ &= \langle C(0, \omega, (j^r_0(x^1))^*), j^r_0(x^1)\rangle = \\ &= \langle C(0, 0, (j^r_0(x^1))^*), j^r_0(x^1)\rangle\,, \end{aligned}$$

where $\omega = (j^r_0((x^2)^r))^*$.

Let us explain $(***)$.

The equality $\langle C(0, \omega, (j^r_0(x^1))^*), j^r_0(x^1)\rangle = \langle C(0, 0, (j^r_0(x^1))^*), j^r_0(x^1)\rangle$ is an immediate consequence of the formula $(*)$.

We prove that $0 = \langle C(((x^2)^r \partial_1)^C{}_{|\omega}), j^r_0(x^1)\rangle$. Let us consider the diffeomorphism $\psi = (x^1 + \frac{1}{r+1}(x^2)^{r+1}, x^2, \dots, x^n)$. Clearly, ψ sends ∂_2

into $\partial_2 + (x^2)^r\partial_1$. It is easily seen that $\det(d_0(\tau_{-\psi(y)} \circ \psi \circ \tau_y)) = 1$ for any $y \in \mathbb{R}^n$ and $j_0^r\psi = \mathrm{id}$. Hence ψ preserves $j_0^r(x^1) \in T_0^{r*,a}\mathbb{R}^n$. Then using the naturality of C with respect to ψ from $(**)$ it follows that $\langle C((\partial_2 + (x^2)^r\partial_1)^C{}_{|\omega}), j_0^r(x^1)\rangle = 0$ for any $\omega \in T_0^{(r),a}\mathbb{R}^n$. Next we apply the linearity of C and $(**)$.

The flow of $(x^2)^r\partial_1$ is $\varphi_t = (x^1 + t(x^2)^r, x^2, \dots, x^n)$ and $\det(d_0(\tau_{-\varphi_t(y)} \circ \varphi_t \circ \tau_y)) = 1$ for any $y \in \mathbb{R}^n$. Then for any $\alpha \in P(r,n)$

$$\begin{aligned}
\langle((x^2)^r\partial_1)^C{}_{|\omega}, j_0^r(x^\alpha)\rangle &= \Big\langle \frac{d}{dt}_{|t=0} T^{(r),a}(\varphi_t)(\omega), j_0^r(x^\alpha)\Big\rangle = \\
&= \frac{d}{dt}_{|t=0} \langle T^{(r),a}(\varphi_t)(\omega), j_0^r(x^\alpha)\rangle = \\
&= \frac{d}{dt}_{|t=0} \langle \omega, j_0^r(x^\alpha \circ \varphi_t)\rangle = \\
&= \Big\langle \omega, j_0^r\Big(\frac{d}{dt}_{|t=0}(x^\alpha \circ \varphi_t)\Big)\Big\rangle = \\
&= \langle \omega, j_0^r((x^2)^r\partial_1 x^\alpha)\rangle = \\
&= \alpha_1\langle \omega, j_0^r((x^2)^r x^{\alpha - e_1})\rangle\,.
\end{aligned}$$

Because of the definition of ω, the last term is equal to 1 if $\alpha = e_1$ and it is equal to 0 in the other cases. Then $((x^2)^r\partial_1)^C{}_{|\omega} = (j_0^r(x^1))^*$ under the isomorphism $V_\omega T^{(r),a}\mathbb{R}^n = T_0^{(r),a}\mathbb{R}^n$. It implies $\langle C((((x^2)^r\partial_1)^C{}_{|\omega}), j_0^r(x^1)\rangle = \langle C(0, \omega, (j_0^r(x^1))^*), j_0^r(x^1)\rangle$. ∎

3 – The tangent map $T\pi : TT^{(r),a}M \to TM$ of the bundle projection $\pi : T^{(r),a}M \to M$ defines a natural transformation over n-manifolds

Proposition 2. *For natural numbers r and n and for a real number $a < 0$ every natural transformation $B : TT^{(r),a} \to T$ over n-manifolds is proportional* (*by a real number*) *to $T\pi$.*

Proof. Clearly, any natural transformation B as in the proposition is uniquely determined by the contractions $\langle B(u), d_0x^1\rangle$ for $u = (u_1, u_2, u_3) \in (TT^{(r),a})_0\mathbb{R}^n = \mathbb{R}^n \times T_0^{(r),a}\mathbb{R}^n \times T_0^{(r),a}\mathbb{R}^n$. Using the invariance of B with respect to the homotheties $a_t = (tx^1, \dots, tx^n)$ for $t \in \mathbb{R}_+$ and the homogeneous function theorem we deduce (similarly as in the proof of Prop. 1) that $\langle B(u), d_0x^1\rangle$ for $u = (u_1, u_2, u_3) \in (TT^{(r),a})_0\mathbb{R}^n = \mathbb{R}^n \times T_0^{(r),a}\mathbb{R}^n \times T_0^{(r),a}\mathbb{R}^n$ is the linear combination (with real coefficients) of the $u_1^1, \dots, u_1^n$ and it is independent of u_2 and u_3, where $u_1 = (u_1^1, \dots, u_1^n) \in \mathbb{R}^n$. Next, using the invariance of B with respect to the homotheties $b_t = (x^1, tx^2, \dots, tx^n)$ we see that $\langle B(u), d_0x^1\rangle$ is proportional (by a real number) to $u_1^1 = \langle T\pi(u), d_0x^1\rangle$. ∎

4 – The first main result of this paper can be written as follows.

THEOREM 1. *For integers $r \geq 1$ and $n \geq 2$ and a negative real number $a < 0$ every natural affinor A on the generalized higher order tangent bundle $T^{(r),a}$ over n-manifolds is proportional* (*by a real number*) *to the identity affinor.*

PROOF. Let A be a natural affinor on $T^{(r),a}M$. Then the composition $T\pi \circ A : TT^{(r),a}M \to TM$ is a natural transformation. By Proposition 2, there exists the real number λ such that $T\pi \circ A = \lambda T\pi$. Then $A - \lambda\,\mathrm{id} : TT^{(r),a}M \to VT^{(r),a}M = T^{(r),a}M \times_M T^{(r),a}M$. Composing this natural transformation with the projection pr_2 onto second factor we obtain a linear natural transformation $\bar{A} = \mathrm{pr}_2 \circ (A - \lambda\,\mathrm{id}) : TT^{(r),a}M \to T^{(r),a}M$. By Proposition 1, $\bar{A} = 0$. Then $A - \lambda\,\mathrm{id} = 0$. The proof of Theorem 1 is complete. ∎

5 – In this section we study linear natural transformations $C : TE^{(r),a} \to E^{(r),a}$ over n-manifolds.

PROPOSITION 3. *For natural numbers r and $n \geq 2$ and a real number $a < 0$ every linear natural transformation $C : TE^{(r),a} \to E^{(r),a}$ over n-manifolds is* 0.

PROOF. From now on the set of all $\alpha \in (\mathbb{N} \cup \{0\})^n$ with $|\alpha| \leq r$ will be denoted by $\overline{P}(r, n)$.

Every section of $T^{(0,0),a}\mathbb{R}^n$ is a real valued function f on $\mathbb{R}^n$ satisfying the transformation rule, see Section 2. Then any element v from the fibre $E_0^{(r),a}\mathbb{R}^n$ is a linear combination of the $(j_0^r x^\alpha)^*$ for all $\alpha \in \overline{P}(r, n)$, where the $(j_0^r x^\alpha)^*$ form the basis dual to the basis $j_0^r x^\alpha \in (J^r T^{(0,0),a})_0\mathbb{R}^n$. From now on we denote the coefficient of v corresponding to $(j_0^r x^\alpha)^*$ by $[v]_\alpha$.

Similarly as in the proof of Proposition 1, any natural transformation C is determined by the contractions $\langle C(u), j_0^r(x^1)\rangle \in \mathbb{R}$ and $\langle C(u), j_0^r(1)\rangle \in \mathbb{R}$ for $u \in (TE^{(r),a})_0\mathbb{R}^n = \mathbb{R}^n \times (VE^{(r),a})_0\mathbb{R}^n = \mathbb{R}^n \times E_0^{(r),a}\mathbb{R}^n \times E_0^{(r),a}\mathbb{R}^n$, $j_0^r(x^1) \in (J^r T^{(0,0),a})_0\mathbb{R}^n$, $j_0^r(1) \in (J^r T^{(0,0),a})_0\mathbb{R}^n$.

For any $k = 1, \ldots, r$ the local diffeomorphism $\psi_k = (x^1 + (x^1)^{k+1}, x^2, \ldots, x^n)$ sends $j_0^r(1) \in (J^r T^{(0,0),a})_0\mathbb{R}^n$ into $j_0^r(1) + a(k+1)j_0^r((x^1)^k) + \ldots$ where the dots is the linear combination of $j_0^r((x^1)^{k+1}), \ldots, j_0^r((x^1)^r)$. By the naturality of C with respect to ψ_τ, the values $\langle C(u), j_0^r(1) + (k+1)aj_0^r((x^1)^k) + \ldots\rangle \in \mathbb{R}$ for $u \in (TE^{(r),a})_0\mathbb{R}^n$ are determined by the values $\langle C(u), j_0^r(1)\rangle \in \mathbb{R}$ for $u \in (TE^{(r),a})_0\mathbb{R}^n$. Hence (since $a \neq 0$) the values $\langle C(u), j_0^r((x^1)^k)\rangle \in \mathbb{R}$ for $u \in (TE^{(r),a})_0\mathbb{R}^n$ are fully determined by the values $\langle C(u), j_0^r(1)\rangle \in \mathbb{R}$ for $u \in (TE^{(r),a})_0\mathbb{R}^n$ (we use the (inverse) induction with respect to k). In particular the values $\langle C(u), j_0^r(x^1)\rangle \in \mathbb{R}$ for $u \in (TE^{(r),a})_0\mathbb{R}^n$ are fully determined by the values $\langle C(u), j_0^r(1)\rangle \in \mathbb{R}$ for $u \in (TE^{(r),a})_0\mathbb{R}^n$.

Therefore C is fully determined by the values $\langle C(u), j_0^r(1)\rangle \in \mathbb{R}$ for $u \in (TE^{(r),a})_0\mathbb{R}^n = \mathbb{R}^n \times E_0^{(r),a}\mathbb{R}^n \times E_0^{(r),a}\mathbb{R}^n$, $j_0^r(1) \in (J^r T^{(0,0),a})_0\mathbb{R}^n$.

We continue the proof of Proposition 3. Similarly as in the proof of Proposition 1, using the homogeneous function theorem, the naturality of C with respect to the $a_t = (tx^1, \ldots, tx^n)$ for $t \in \mathbb{R}_+$, the linearity of C and the assumption $a < 0$ we deduce that given $u = (u_1, u_2, u_3) \in (TE^{(r),a})_0\mathbb{R}^n = \mathbb{R}^n \times E_0^{(r),a}\mathbb{R}^n \times E_0^{(r),a}\mathbb{R}^n$, $u_1 = (u_1^1, \ldots, u_1^n) \in \mathbb{R}^n$, $u_2, u_3 \in E_0^{(r),a}\mathbb{R}^n$ we have $\langle C(u), j_0^r(1)\rangle = \sum_{i=1}^n \rho_i u_1^i + \mu[u_3]_{(0)}$, where ρ_i and μ are the real numbers. Next, using the invariance of C with respect to the $b_{t,i} = (x^1, \ldots, tx^i, \ldots, x^n)$ (only i-th position is exceptional) we deduce that $\rho_i = 0$ for $i = 1, \ldots, n$, if $a \neq -1$. (For, $b_{t,i}$ sends $(j_0^r(x^\alpha))^*$ into $t^{-a+\alpha_i}(j_0^r(x^\alpha))^*$.) If $a = -1$, then (since $n \geq 2$) using the invariance of C with respect to the a_t we deduce that $\rho_i = 0$ for $i = 1, \ldots, n$. Then

$$(*)' \qquad \langle C(u), j_0^r(1)\rangle = \mu[u_3]_{(0)}\ ,$$

where μ is the real number. In particular

$$(**)' \qquad \langle C(\partial^C_{1\ |\omega}), j_0^r(1)\rangle = \langle C(e_1, \omega, 0), j_0^r(1)\rangle = 0$$

for any $\omega \in E_0^{(r),a}\mathbb{R}^n$, where $(\ )^C$ is the complete lift of vector fields to $E^{(r),a}$.

Clearly, the proof of the proposition will be complete after proving that $\mu = 0$, i.e. $\langle C(0,0,(j_0^r(1))^*), j_0^r(1)\rangle = 0$. But we have

$$
(***)' \qquad
\begin{aligned}
0 &= \langle C(((x^1)^{r+1}\partial_1)^C{}_{|\omega}), j_0^r(1)\rangle = \\
&= \langle < C(0,\omega,(j_0^r(1))^* + \dots), j_0^r(1)\rangle = \\
&= \langle C(0,0,(j_0^r(1))^*), j_0^r(1)\rangle\,,
\end{aligned}
$$

where $\omega = \frac{1}{-a(r+1)}(j_0^r((x^1)^r))^*$ and where the dots denote the linear combination with real coefficients of the $(j_0^r(x^\alpha))^*$ for $\alpha \in \overline{P}(r,n)$ with $|\alpha| \geq 1$.

Let us explain $(***)'$.

The equality $\langle C(0,\omega,(j_0^r(1))^*+\dots), j_0^r(1)\rangle = \langle C(0,0,(j_0^r(1))^*), j_0^r(1)\rangle$ is an immediate consequence of the formula $(*)'$.

We prove that $0 = \langle C(((x^1)^{r+1}\partial_1)^C{}_{|\omega}), j_0^r(1)\rangle$. The vector fields ∂_1 and $\partial_1 + (x^1)^{r+1}\partial_1$ have the same r-jets at $0 \in \mathbb{R}^n$. Hence by the result of Zajtz [5], there exists a diffeomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ with $j_0^{r+1}(\psi) = \mathrm{id}$ sending ∂_1 into $\partial_1 + (x^1)^{r+1}\partial_1$ near 0. Clearly, the natural bundle $J^rT^{(0,0),a}$ is of order $\leq r+1$. Hence ψ preserves $j_0^r(1) \in (J^rT^{(0,0),a})_0\mathbb{R}^n$ because of the jets argument. Now, by the naturality of C with respect to ψ from $(**)'$ it follows that $\langle C((\partial_1 + (x^1)^{r+1}\partial_1)^C{}_{|\omega}), j_0^r(1)\rangle = 0$ for any $\omega \in E_0^{(r),a}\mathbb{R}^n$. Next we use the linearity of C and $(**)'$.

Clearly, the flow of $(x^1)^{r+1}\partial_1$ is $\varphi_t = (x^1 + t(x^1)^{r+1} + t^2(\dots), x^2, \dots, x^n)$, where the dots is the smooth function of x^1 and t. For any $t \in \mathbb{R}$ the transformation φ_{-t} sends the Section 1 of $J^rT^{(0,0),a}\mathbb{R}^n$ into $(1 - t(r+1)(x^1)^r + t^2\alpha(t,x^1))^a$ near $0 \in \mathbb{R}^n$, where $\alpha(t,x^1)$ is the smooth function in t and x^1. Since the Taylor expansion at $y=0$ of $(1+y)^a$ is $1+ay+\dots$, we have $j_0^r((1-t(r+1)(x^1)^r+t^2\alpha(t,x^1))^a) = j_0^r(1-at(r+1)(x^1)^r+t^2\psi(t,x^1))$. Then

$$
\begin{aligned}
\langle ((x^1)^{r+1}\partial_1)^C_{|\omega}, j_0^r(1)\rangle &= \Big\langle \frac{d}{dt}_{|t=0} E^{(r),a}(\varphi_t)(\omega), j_0^r(1)\Big\rangle = \\
&= \frac{d}{dt}_{|t=0}\langle E^{(r),a}(\varphi_t)(\omega), j_0^r(1)\rangle = \\
&= \frac{d}{dt}_{|t=0}\langle \omega, j_0^r(1 - at(r+1)(x^1)^r + t^2\psi(t,x^1))\rangle = \\
&= \langle \omega, -a(r+1)j_0^r((x^1)^r)\rangle = 1
\end{aligned}
$$

because of the definition of ω. Then $((x^1)^{r+1}\partial_1)^C{}_{|\omega} = (j^r_0(1))^* + \dots$ under the isomorphism $V_\omega E^{(r),a}\mathbb{R}^n = E^{(r),a}_0\mathbb{R}^n$. It implies $\langle C(((x^1)^{r+1}\partial_1)^C{}_{|\omega}), j^r_0(1)\rangle = \langle C(0, \omega, (j^r_0(1))^* + \dots), j^r_0(1)\rangle$.

6 – The tangent map $T\pi : TE^{(r),a}M \to TM$ of the bundle projection $\pi : E^{(r),a}M \to M$ defines a natural transformation over n-manifolds

Proposition 4. *For natural numbers r and $n \geq 2$ and for a real number $a < 0$ every natural transformation $B : TE^{(r),a} \to T$ over n-manifolds is proportional* (*by a real number*) *to $T\pi$.*

Proof. Clearly, any natural transformation B as in the proposition is uniquely determined by the contractions $\langle B(u), d_0x^1\rangle$ for $u = (u_1, u_2, u_3) \in (TE^{(r),a})_0\mathbb{R}^n = \mathbb{R}^n \times E^{(r),a}_0\mathbb{R}^n \times E^{(r),a}_0\mathbb{R}^n$. Using the invariance of B with respect to the homotheties $a_t = (tx^1, \dots, tx^n)$ for $t \in \mathbb{R}_+$, the homogeneous function theorem and the assumption $a < 0$ we deduce (similarly as in the proof of Prop. 1) that $\langle B(u), d_0x^1\rangle = \sum_1^n \rho_i u^i_1 + w([u_2]_{(0)}, [u_3]_{(0)})$, where ρ_i are the real numbers and w is the polynomial with $w(0,0) = 0$, $u_1 = (u^1_1, \dots, u^n_1) \in \mathbb{R}^n$ and $[v]_\alpha$ is as in the proof of Proposition 3. Next, using the invariance of B with respect to the homotheties $b_t = (x^1, tx^2, \dots, tx^n)$ and the assumption $a \neq 0$ and $n \geq 2$ we see that $\langle B(u), d_0x^1\rangle$ is proportional (by ρ_1) to u^1_1. ∎

7 – The second main result of this paper can be written as follows.

Theorem 2. *For integers $r \geq 1$ and $n \geq 2$ and a real number $a < 0$ every natural affinor A on the extended generalized higher order tangent bundle $E^{(r),a}$ over n-manifolds is proportional* (*by a real number*) *to the identity affinor.*

Proof. Using similar arguments as in Section 3 from Propositions 3 and 4 we obtain Theorem 2. ∎

8 – From Theorems 1 and 2 we obtain immediately.

Corollary 1. *For integers $r \geq 1$ and $n \geq 2$ and a negative real number $a < 0$ there are no natural connections on $T^{(r),a}$ over n-manifolds.*

Corollary 2. *For integers $r \geq 1$ and $n \geq 2$ and a real number $a < 0$ there are no natural connections on $E^{(r),a}$ over n-manifolds.*

9 – Remark. Similarly as $T^{(r),a}$ and $E^{(r),a}$, starting from the action $GL(n, \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ given by $(B, x) \to \operatorname{sgn}(\det(B))|\det(B)|^a x$ instead of $\alpha^{(a)} : GL(n, \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$, we can define natural vector bundles $\tilde{T}^{(r),a}$ and $\tilde{E}^{(r),a}$ over n-manifolds. Clearly, Theorem 1 is true for $\tilde{T}^{(r),a}$ instead of $T^{(r),a}$. Similarly, Theorem 2 is true for $\tilde{E}^{(r),a}$ instead of $E^{(r),a}$. We use the same proofs with $\tilde{T}^{(r),a}$ instead of $T^{(r),a}$ or $\tilde{E}^{(r),a}$ instead of $E^{(r),a}$.

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